

# Some extremal problems for hereditary properties of graphs

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## Abstract

Let  $\mathcal{P}$  be an infinite hereditary property of graphs. Define

$$\pi(\mathcal{P}) = \lim_{n \rightarrow \infty} \binom{n}{2}^{-1} \max\{e(G) : G \in \mathcal{P} \text{ and } v(G) = n\}.$$

In this note  $\pi(\mathcal{P})$  is determined for every hereditary property  $\mathcal{P}$ .

The same problem is studied for a more general parameter  $\lambda^{(\alpha)}(G)$ , defined for every real number  $\alpha \geq 1$  and every graph  $G$  as

$$\lambda^{(\alpha)}(G) = \max_{|x_1|^\alpha + |x_2|^\alpha + \dots + |x_n|^\alpha = 1} 2 \sum_{\{u,v\} \in E(G)} x_u x_v.$$

It is known that the limit

$$\lambda^{(\alpha)}(\mathcal{P}) = \lim_{n \rightarrow \infty} n^{2/\alpha-2} \max\{\lambda^{(\alpha)}(G) : G \in \mathcal{P} \text{ and } v(G) = n\}$$

exists. A key result of the note is the equality

$$\lambda^{(\alpha)}(\mathcal{P}) = \pi(\mathcal{P}),$$

which holds for all  $\alpha > 1$ .

## 1 Introduction

In this note we study problems stemming from the following one:

*What is the maximum number of edges a graph of order  $n$ , belonging to some hereditary property  $\mathcal{P}$ .*

Let us recall that a hereditary property is a family of graphs closed under taking induced subgraphs. For example, given a set of graphs  $\mathcal{F}$ , the family of all graphs that do not contain any  $F \in \mathcal{F}$  as an induced subgraph is a hereditary property, denoted as  $Her(\mathcal{F})$ .

It seems that the above classically shaped problem has been disregarded in the rich literature on hereditary properties, so we fill in this gap below.

Writing  $\mathcal{P}_n$  for the set of all graphs of order  $n$  in a property  $\mathcal{P}$ , our problem now reads as: *Given a hereditary property  $\mathcal{P}$ , find*

$$ex(\mathcal{P}, n) = \max_{G \in \mathcal{P}_n} e(G). \quad (1)$$

Finding  $ex(\mathcal{P}, n)$  exactly seems hopeless for arbitrary  $\mathcal{P}$ . A more feasible approach has been suggested by Katona, Nemetz and Simonovits in [7] who proved the following fact:

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**Proposition 1** *If  $\mathcal{P}$  is a hereditary property, then the sequence*

$$\left\{ ex(\mathcal{P}, n) \binom{n}{2}^{-1} \right\}_{n=1}^{\infty}$$

*is nonincreasing and so the limit*

$$\pi(\mathcal{P}) = \lim_{n \rightarrow \infty} ex(\mathcal{P}, n) \binom{n}{2}^{-1}$$

*always exists.*

One of the aims of this paper is to establish  $\pi(\mathcal{P})$  for every  $\mathcal{P}$ , but our main interest is in extremal problems about a different graph parameter, denoted by  $\lambda^{(\alpha)}(G)$  and defined as follows: *for every graph  $G$  and every real number  $\alpha \geq 1$ , let*

$$\lambda^{(\alpha)}(G) = \max_{|x_1|^\alpha + \dots + |x_n|^\alpha = 1} 2 \sum_{\{u,v\} \in E(G)} x_u x_v.$$

Note first that  $\lambda^{(2)}(G)$  is the well-studied spectral radius of  $G$ , and second, that  $\lambda^{(1)}(G)$  is another much studied parameter, known as the Lagrangian of  $G$ . So  $\lambda^{(\alpha)}(G)$  is a common generalization of two parameters that have been widely used in extremal graph theory.

The parameter  $\lambda^{(\alpha)}(G)$  has been recently introduced and studied for uniform hypergraphs first, by Keevash, Lenz and Mubayi in [6] and next by the author, in [13]. Here we shall study  $\lambda^{(\alpha)}(G)$  in the same setting as the number of edges in (1). Thus, given a hereditary property  $\mathcal{P}$ , set

$$\lambda^{(\alpha)}(\mathcal{P}, n) = \max_{G \in \mathcal{P}_n} \lambda^{(\alpha)}(G). \quad (2)$$

As with  $ex(\mathcal{P}, n)$  finding  $\lambda^{(\alpha)}(\mathcal{P}, n)$  seems hopeless for arbitrary  $\mathcal{P}$ . So, to begin with, the following theorem has been proved in [13] as an analog to Proposition 1.

**Theorem 2** *Let  $\alpha \geq 1$ . If  $\mathcal{P}$  is a hereditary property, then the limit*

$$\lambda^{(\alpha)}(\mathcal{P}) = \lim_{n \rightarrow \infty} \lambda^{(\alpha)}(\mathcal{P}, n) n^{(2/\alpha)-2} \quad (3)$$

*exists.*

Thus, a natural question is to find  $\lambda^{(\alpha)}(\mathcal{P})$  for every  $\mathcal{P}$  and every  $\alpha \geq 1$ . The main goal of this note to answer this question completely.

It turns out that  $\lambda^{(\alpha)}(\mathcal{P})$  and  $\pi(\mathcal{P})$  are closely related. For example, results proved in [13] imply that  $\lambda^{(\alpha)}(\mathcal{P}) \geq \pi(\mathcal{P})$  for every  $\mathcal{P}$  and every  $\alpha \geq 1$ , moreover, if  $\alpha \geq 2$ , then  $\lambda^{(\alpha)}(\mathcal{P}) = \pi(\mathcal{P})$ . In this note we shall extend this relation to: *if  $\alpha > 1$ , then  $\lambda^{(\alpha)}(\mathcal{P}) = \pi(\mathcal{P})$ .*

## 2 Main results

For notation and concepts undefined here, the reader is referred to [1].

Note first that every hereditary property  $\mathcal{P}$  is trivially characterized by  $\mathcal{P} = \text{Her}(\overline{\mathcal{P}})$ , where  $\overline{\mathcal{P}}$  is the family of all graphs that are not in  $\mathcal{P}$ ; however, typically  $\mathcal{P}$  can be given as  $\mathcal{P} = \text{Her}(\mathcal{F})$  for some  $\mathcal{F}$  that is only a small fraction of  $\overline{\mathcal{P}}$ .

Recall next that a complete  $r$ -partite graph is a graph whose vertices are split into  $r$  nonempty independent sets so that all edges between vertices of different classes are present. In particular, a 1-partite graph is just a set of independent vertices.

To characterize  $\pi(\mathcal{P})$  and  $\lambda^{(\alpha)}(\mathcal{P})$  we shall need two numeric parameters defined for every family of graphs  $\mathcal{F}$ . First, let

$$\underline{\omega}(\mathcal{F}) = \begin{cases} 0, & \text{if } \mathcal{F} \text{ contains no cliques;} \\ \min\{r : K_r \in \mathcal{F}\}, & \text{otherwise,} \end{cases}$$

and second, let

$$\beta(\mathcal{F}) = \begin{cases} 0, & \text{if } \mathcal{F} \text{ contains no complete partite graphs;} \\ \min\{r : \mathcal{F} \text{ contains a complete } r\text{-partite graph}\}, & \text{otherwise.} \end{cases}$$

The parameters  $\underline{\omega}(\mathcal{F})$  and  $\beta(\mathcal{F})$  are quite informative about the hereditary property  $Her(\mathcal{F})$ , as seen first in the following observation.

**Proposition 3** *If the property  $\mathcal{P} = Her(\mathcal{F})$  is infinite, then  $\underline{\omega}(\mathcal{F}) = 0$  or  $\underline{\omega}(\mathcal{F}) \geq 2$  and  $\beta(\mathcal{F}) \geq 2$ .*

**Proof** Suppose that  $\underline{\omega}(\mathcal{F}) \neq 0$ . If  $\underline{\omega}(\mathcal{F}) = 1$ , then  $\mathcal{P}$  is empty, so we can suppose that  $\underline{\omega}(\mathcal{F}) \geq 2$ . This implies that  $\beta(\mathcal{F}) > 0$ , as  $\mathcal{F}$  contains  $K_r$  for some  $r \geq 2$  and  $K_r$  is a complete  $r$ -partite graph. If  $\beta(\mathcal{F}) = 1$ , then  $\mathcal{F}$  contains a graph  $G$  consisting of isolated vertices, say  $G$  is on  $s$  vertices. If  $\mathcal{P}$  is infinite, choose a member  $G \in \mathcal{P}$  with  $v(G) \geq r(K_r, K_s)$ , where  $r(K_r, K_s)$  is the Ramsey number of  $K_r$  vs.  $K_s$ . Then either  $G$  contains a  $K_r$  or an independent set on  $s$  vertices, both of which are forbidden. It turns out that  $\beta(\mathcal{F}) \geq 2$ , proving Proposition 3.  $\square$

Clearly the study of (1) and (2) makes sense only if  $\mathcal{P}$  is infinite and Proposition 3 provides necessary condition for this property of  $\mathcal{P}$ . The following theorem completely characterizes  $\pi(\mathcal{P})$ .

**Theorem 4** *Let  $\mathcal{F}$  be a family of graphs. If the property  $\mathcal{P} = Her(\mathcal{F})$  is infinite, then*

$$\pi(\mathcal{P}) = \begin{cases} 1, & \text{if } \underline{\omega}(\mathcal{F}) = 0; \\ 1 - \frac{1}{\beta(\mathcal{F})-1}, & \text{otherwise.} \end{cases}.$$

**Proof** Indeed, since  $\mathcal{P}$  is infinite, Proposition 3 implies that  $\underline{\omega}(\mathcal{F}) = 0$  or  $\underline{\omega}(\mathcal{F}) \geq 2$  and  $\beta(\mathcal{F}) \geq 2$ . If  $\underline{\omega}(\mathcal{F}) = 0$ , then  $K_n \in \mathcal{P}_n$ , because all subgraphs of  $K_n$  are complete and do not belong to  $\mathcal{F}$ . Therefore,

$$ex(\mathcal{P}, n) = \binom{n}{2},$$

and so,  $\pi(\mathcal{P}) = 1$ . Assume that  $\underline{\omega}(\mathcal{F}) \geq 2$  and  $\beta(\mathcal{F}) \geq 2$ , and set for short  $r = \underline{\omega}(\mathcal{F}) \geq 2$  and  $\beta = \beta(\mathcal{F})$ . Next, we shall prove that  $T_{\beta-1}(n) \in \mathcal{P}_n$ , where  $T_{\beta-1}(n)$  is the complete  $(\beta-1)$ -partite Turán graph of order  $n$ . Indeed all subgraphs of  $T_{\beta-1}(n)$  are complete  $r$ -partite graphs for some  $r \leq \beta-1$ , so should one of them belong to  $\mathcal{F}$ , we would have  $\beta(\mathcal{F}) \leq \beta-1 = \beta(\mathcal{F})-1$ , a contradiction. Therefore,

$$ex(\mathcal{P}, n) \geq e(T_{\beta-1}(n)) = \left(1 - \frac{1}{\beta-1} + o(1)\right) \binom{n}{2},$$

and so

$$\pi(\mathcal{P}) \geq 1 - \frac{1}{\beta(\mathcal{F})-1}.$$

To finish the proof we shall prove the opposite inequality. Let  $F \in \mathcal{F}$  be a complete  $\beta$ -partite graph, known to exist by the definition of  $\beta(\mathcal{F})$  and let  $s$  be the maximum of the sizes of its vertex classes.

Now assume that  $\varepsilon > 0$  and set  $t = r(K_r, K_s)$ , where  $r(K_r, K_s)$  is the Ramsey number of  $K_r$  vs.  $K_s$ . If  $n$  is large enough and  $G \in \mathcal{P}_n$  satisfies

$$e(G) > \left(1 - \frac{1}{\beta(\mathcal{F}) - 1} + \varepsilon\right) \binom{n}{2},$$

then by the theorem of Erdős and Stone [5],  $G$  contains a subgraph  $G_0 = K_\beta(t)$ , that is to say, a complete  $\beta$ -partite graph with  $t$  vertices in each vertex class. Since  $K_r \in \mathcal{F}$ , we see that  $G_0$  contains no  $K_r$ , hence each vertex class of  $G_0$  contains an independent set of size  $s$ , and so  $G$  contains an induced subgraph  $K_\beta(s)$ , which in turn contains an induced copy of  $F$ . Hence, if  $n$  is large enough and  $G \in \mathcal{P}_n$ , then

$$e(G) \binom{n}{2}^{-1} \leq 1 - \frac{1}{\beta(\mathcal{F}) - 1} + \varepsilon.$$

This inequality implies that

$$\pi(\mathcal{P}) \leq 1 - \frac{1}{\beta(\mathcal{F}) - 1},$$

completing the proof.  $\square$

We continue now with establishing  $\lambda^{(\alpha)}(\mathcal{P})$  for  $\alpha > 1$ . The proof of our key Theorem 7 relies on several other results, some of which are stated within the proof itself. We give two other before the theorem. The first one follows from a result in [13], but for reader's sake we reproduce its short proof here.

**Theorem 5** *Let  $\alpha \geq 1$ . If  $G$  is a graph with  $m$  edges and  $n$  vertices, with no  $K_{r+1}$ , then*

$$\lambda^{(\alpha)}(G) \leq \left(1 - \frac{1}{r}\right)^{1/\alpha} (2m)^{1-1/\alpha} \quad (4)$$

and

$$\lambda^{(\alpha)}(G) \leq \left(1 - \frac{1}{r}\right) n^{2-2/\alpha}. \quad (5)$$

**Proof** Indeed, let  $\mathbf{x} = (x_1, \dots, x_n)$  be a vector such that  $|x_1|^\alpha + \dots + |x_n|^\alpha = 1$  and

$$\lambda^{(\alpha)}(G) = 2 \sum_{\{u,v\} \in E(G)} x_u x_v.$$

Applying Jensen's inequality, we see that

$$\begin{aligned} \lambda^{(\alpha)}(G) &= 2 \sum_{\{u,v\} \in E(G)} x_u x_v \leq 2 \sum_{\{u,v\} \in E(G)} |x_u| |x_v| \\ &\leq (2m)^{1-1/\alpha} \left( 2 \sum_{\{u,v\} \in E(G)} |x_u|^\alpha |x_v|^\alpha \right)^{1/\alpha}. \end{aligned}$$

But by the result of Motzkin and Straus [8], we have

$$2 \sum_{\{u,v\} \in E(G)} |x_u|^\alpha |x_v|^\alpha \leq 1 - \frac{1}{r},$$

and inequality (4) follows. Now inequality (5) follows from (4) by Turán's theorem  $2m < (1 - 1/r) n^2$ .  $\square$

We shall need also the following proposition (Proposition 29, [13]) whose proof we omit.

**Proposition 6** *Let  $\alpha \leq 1$ ,  $k > 1$  and  $G_1$  and  $G_2$  be graphs on the same vertex set. If  $G_1$  and  $G_2$  differ in at most  $k$  edges, then*

$$\left| \lambda^{(\alpha)}(G_1) - \lambda^{(\alpha)}(G_2) \right| \leq (2k)^{1-1/\alpha}.$$

Here is the main theorem about  $\lambda^{(\alpha)}(\mathcal{P})$ .

**Theorem 7** *Let  $\alpha > 1$  and let  $\mathcal{F}$  be a family of graphs. If the property  $\mathcal{P} = \text{Her}(\mathcal{F})$  is infinite, then*

$$\lambda^{(\alpha)}(\mathcal{P}) = \begin{cases} 1, & \text{if } \underline{\omega}(\mathcal{F}) = 0; \\ 1 - \frac{1}{\beta(\mathcal{F})-1}, & \text{otherwise.} \end{cases}.$$

**Proof** First note the inequality

$$\lambda^{(\alpha)}(G) \geq 2e(G)/n^{2/\alpha},$$

which follows by taking  $(x_1, \dots, x_n) = (n^{-1/\alpha}, \dots, n^{-1/\alpha})$  in (2). So we see that

$$\lambda^{(\alpha)}(\mathcal{P}) \geq \pi(\mathcal{P}),$$

and this, together with Theorem 4 gives  $\lambda^{(\alpha)}(\mathcal{P}) = 1$  if  $\underline{\omega}(\mathcal{F}) = 0$  and

$$\lambda^{(\alpha)}(\mathcal{P}) \geq 1 - \frac{1}{\beta(\mathcal{F})-1}$$

otherwise. To finish the proof we shall prove that

$$\lambda^{(\alpha)}(\mathcal{P}) \leq 1 - \frac{1}{\beta(\mathcal{F})-1}$$

For the purposes of this proof, write  $k_r(G)$  for the number of  $r$ -cliques of  $G$ . Let  $F \in \mathcal{F}$  be a complete  $\beta$ -partite graph, which exists by the definition of  $\beta(\mathcal{F})$ , and let  $s$  be the maximum of the sizes of its vertex classes.

We recall the following particular version of the Removal Lemma, one of the important consequences of the Szemerédi Regularity Lemma ([15],[1]):

**Removal Lemma** *Let  $r \geq 2$  and  $\varepsilon > 0$ . There exists  $\delta = \delta(r, \varepsilon) > 0$  such that if  $G$  is a graph of order  $n$ , with  $k_r(G) < \delta n^r$ , then there is a graph  $G_0 \subset G$  such that  $e(G_0) \geq e(G) - \varepsilon n^2$  and  $k_r(G_0) = 0$ .*

In [11] we have proved the following theorem:

**Theorem A** *For all  $r \geq 2$ , and  $\varepsilon > 0$  there exists  $\delta = \delta(r, \varepsilon) > 0$  such that if  $G$  a graph of order  $n$  with  $k_r(G) > \varepsilon n^r$ , then  $G$  contains a  $K_r(s)$  with  $s = \lfloor \delta \log n \rfloor$ .*

Now let  $\varepsilon > 0$ , choose  $\delta = \delta(\beta, \varepsilon)$  as in the Removal Lemma, and set  $t = r(K_r, K_s)$ , where  $r(K_r, K_s)$  is the Ramsey number of  $K_r$  vs.  $K_s$ . If  $G \in \mathcal{P}_n$ , then  $K_\beta(t) \not\subseteq G$  as otherwise we see as in proof of Theorem 4 that  $G$  contains an induced copy of  $F$ . So by Theorem A, if  $n$  is large enough, then  $k_\beta(G) \leq \delta n^r$ . Now by the Removal Lemma there is a graph  $G_0 \subset G$  such that  $e(G_0) \geq e(G) - \varepsilon n^2$  and  $k_\beta(G_0) = 0$ .

By Propositions 6 and 5, for  $n$  sufficiently large, we see that

$$\lambda^{(\alpha)}(G) \leq \lambda^{(\alpha)}(G_0) + (2\varepsilon n)^{2-2/\alpha} \leq \left(1 - \frac{1}{\beta-1}\right) n^{2-2/\alpha} + (2\varepsilon n)^{2-2/\alpha},$$

and hence,

$$\lambda^{(\alpha)}(\mathcal{P}, n) n^{2/\alpha-2} \leq 1 - \frac{1}{\beta-1} + (2\varepsilon)^{2-2/\alpha}$$

Since  $\varepsilon$  can be made arbitrarily small, we see that

$$\lambda^{(\alpha)}(\mathcal{P}) \leq 1 - \frac{1}{\beta-1},$$

completing the proof of Theorem 7. □

To complete the picture, we need to determine the dependence of  $\lambda^{(1)}(\mathcal{P})$  on  $\mathcal{P}$ . Using the well-known idea of Motzkin and Straus, we come up with the following theorem, whose proof we omit

**Theorem 8**  $\lambda^{(1)}(\mathcal{P})$  Let  $\mathcal{P}$  be an infinite hereditary property. Then  $\lambda^{(1)}(\mathcal{P}) = 1$  if  $\mathcal{P}$  contains arbitrary large cliques, or  $\lambda^{(1)}(\mathcal{P}) = 1 - 1/r$ , where  $r$  is the size of the largest clique in  $\mathcal{P}$ .

### 3 Concluding remarks

In a cycle of papers the author has shown that many classical extremal results like the Erdős-Stone-Bollobás theorem [2], the Stability Theorem of Erdős [3, 4] and Simonovits [14], and various saturation problems can be strengthened by recasting them for the largest eigenvalue instead of the number of edges; see [12] for overview and references.

The results in the present note and in [13] show that some of these results can be extended further for  $\lambda^{(\alpha)}(G)$  and  $\alpha \geq 1$ . A natural challenge here is to reprove systematically all of the above problems by substituting  $\lambda^{(\alpha)}(G)$  for the number of edges.

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